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NESTED BOUNDS FOR SOLUTIONS OF DIFFERENTIAL EQUATIONS

by

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APPROVED FOR DISTRIBUTION

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ABSTRACT

For a particular class of ordinary differential equations, an iterative procedure is described that gives a sequence of nested pairs of lower and upper bounds for the solution. Simple conditions are found under which the bounds converge to the solution. The method is used to study equations of Lanchester's type, which have applications to the deterministic description of situations in naval warfare.

INTRODUCTION

The dynamics of combat in general, and situations in naval warfare in particular, can be described deterministically by means of differential equations usually called Lanchester equations, Lanchester being among the first to investigate such descriptions of combat [Ref. 1].

For those equations to provide useful mathematical models, it is desirable that information concerning the qualitative behaviour of the solutions can be obtained without too much effort. Indeed, even if particular solutions can always be found by numerical integration, it is of more fundamental interest to be able to predict, for instance, the effect of parameters and initial conditions on the final result of the combat or on the value of the variables at a given time.

The qualitative behaviour of the solutions is studied here through the determination of successive lower and upper bounds. The first steps of the iteration, obtained by hand computation, will provide bounds for the variables (which might represent, for instance, force levels) that depend explicitly on the parameters. It is then easy to find, for instance, bounds for the time required for enemy or own forces to fall below a given value.

Besides its usefulness in a qualitative analysis, the procedure developed here can sometimes be adapted for implementation on digital computers in order to determine, by rapidly converging approximations, the exact solution. This is justified when the successive linear differential equations can be replaced by successive algebraic equations; an example of such a situation is given below.

Although this work has been inspired by the consideration of equations of Lanchester's type, the results are by no means restricted to such equations, as appears clearly from the hypotheses made on the equations. On the other hand, it must be mentioned that this note is not concerned with the elaboration of models. For these, the reader is referred, for example, to B.W. Conolly [Ref. 2] and to K.M. Mjelde and R.R. Wiederkehr [Ref. 3], who analyse situations related to the defence of merchant shipping against submarines. Although the work of K.M. Mjelde and R.R. Wiederkehr is also concerned with the study of the qualitative behaviour of the solutions of Lanchester equations, its emphasis differs somewhat from our own, which gives greater weight to the mathematical aspects of the problem.

1. STATEMENT OF THE PROBLEM AND NOTATIONS

Let us consider a system of differential equations in vector notation:

$$\dot{\underline{y}} = A\underline{y} + \underline{Y}(t,\underline{y}) \quad , \tag{1}$$

$\underline{y}, \underline{Y} \in E_n$ (the n-dimensional euclidian space), A is an $n \times n$ matrix, $\dot{\underline{y}}$ is the derivative of $\underline{y}$ with respect to t.

Systems of type [1] will be investigated in this work, with the important restriction that the elements a_{ij} of A will be assumed to be non-negative, except those on the main diagonal, which can be of either sign: $a_{ij} \geq 0$ $(i \neq j)$. If a given system does not satisfy those conditions, an appropriate transformation might force it to do so. A class of particularly simple transformations of that kind is considered in Appendix A. Another possibility would be to include in $\underline{Y}(t,\underline{y})$ the linear terms a_{ij} y_j having an unacceptable sign. However, this will in general make our method less efficient.

As vectors in E_n will often be compared, it is useful to introduce the following notations, for $\underline{x},\underline{y},\underline{z} \in E_n$

$\underline{y} < \underline{z}$	if	$y_i < z_i$	$(1 \leq i \leq n)$,
$\underline{y} \leq \underline{z}$	if	$y_i \leq z_i$	$(1 \leq i \leq n)$,
$\underline{y} > \underline{x}$	if	$y_i > x_i$	$(1 \leq i \leq n)$,
$\underline{y} \geq \underline{x}$	if	$y_i \geq x_i$	$(1 \leq i \leq n)$.

2. RELATIONS BETWEEN SOLUTIONS OF DIFFERENTIAL EQUATIONS

Theorem 1

If $a_{ij} \geq 0 \ (i \neq j)$ and if $\underline{G}(t)$ is any function satisfying the relation

$$\underline{Y}[t,\underline{y}(t)] < \underline{G}(t) \qquad \text{for} \quad t \geq 0, \tag{2}$$

the solutions of the differential equations

$$\dot{\underline{y}} = A\underline{y} + \underline{Y}(t,\underline{y}) \ , \quad \underline{y}(0) = \underline{a} \tag{3}$$

$$\dot{\underline{z}} = A\underline{z} + \underline{G}(t) \quad , \quad \underline{z}(0) = \underline{b} > \underline{a} \tag{4}$$

are such that

$$\underline{y}(t) < \underline{z}(t) \qquad \text{for} \quad t \geq 0. \tag{5}$$

PROOF. Let us consider the set S_i of values $t \geq 0$ for which $y_i(t) \geq z_i(t)$ and let $S = \bigcup_{1 \leq i \leq n} S_i$.

If S is not void, it has an infimum $\xi > 0$ and

a. $\underline{y}(t) < \underline{z}(t)$ for $0 \leq t < \xi$,

b. $\exists i \ : \ y_i(\xi) = z_i(\xi)$.

Then,

$$\dot{y}_i(\xi) = (A\underline{y})_i + Y_i(\xi,\underline{y}) < (A\underline{z})_i + G_i(\xi) = \dot{z}_i(\xi)$$

and a positive number δ exists, such that

$$\frac{y_i(t) - y_i(\xi)}{t - \xi} < \frac{z_i(t) - z_i(\xi)}{t - \xi} \quad \text{for} \quad \xi - \delta \le t < \xi \quad .$$

Consequently,

$$y_i(t) > z_i(t) \quad \text{for} \quad \xi - \delta \le t < \xi \quad ;$$

the contradiction with [6] shows that S must be void.

<u>Corollary 1</u>

If the condition $\underline{b} > \underline{a}$ is replaced by the less stringent condition $\underline{b} \ge \underline{a}$, a similar result is obtained, conclusion [5] being replaced by

$$\underline{y}(t) < \underline{z}(t) \quad \text{for} \quad t > 0 \quad . \qquad [7]$$

PROOF. If $a_i = b_i$ for some $i(1 \le i \le n)$,

$$\dot{y}_i(0) = (A\underline{a})_i + Y_i(0,\underline{a}) < (A\underline{b})_i + G_i(0) = \dot{z}_i(0) \quad .$$

Consequently, for a sufficiently small positive number ϵ_i, the relation $y_i(t) < z_i(t)$ is valid for $0 < t \le \epsilon_i$. More generally, a positive number ϵ exists such that

$$\underline{y}(t) < \underline{z}(t) \quad \text{for} \quad 0 < t \le \epsilon \quad .$$

Then, by Theorem 1,

$$\underline{y}(t) < \underline{z}(t) \quad \text{for} \quad t \ge \epsilon \quad .$$

and Corollary 1 is proved.

Corollary 2

If condition [2] is replaced by

$$\underline{Y}[t,\underline{y}(t)] \le \underline{G}(t) \qquad \text{for} \quad t \ge 0 \quad , \tag{8}$$

conclusion [7] must be replaced by

$$\underline{y}(t) \le \underline{z}(t) \qquad \text{for} \quad t \ge 0 \quad . \tag{9}$$

PROOF. Let ϵ_n be a monotonically-decreasing sequence tending toward zero. Because of Corollary 1, the solution $\underline{z}^{(n)}(t)$ of the equation

$$\dot{\underline{z}}^{(n)} = A\,\underline{z}^{(n)} + \underline{G}(t) + \epsilon_n \underline{h} \quad , \qquad \underline{z}^{(n)}(0) = \underline{a} \quad ,$$

where all the components of $\underline{h}$ are equal to 1, is such that

$$\underline{y}(t) \le \underline{z}^{(n)}(t) \qquad \text{for} \quad t \ge 0 \quad .$$

By the continuity of the solution with respect to a parameter, the relation

$$\underline{y}(t) \le \underline{z}(t) \qquad \text{for} \quad t \ge 0 \quad .$$

is established for the solution $\underline{z}(t)$ of

$$\dot{\underline{z}} = A\underline{z} + \underline{G}(t) \; , \qquad \underline{z}(0) = \underline{a}.$$

If lower bounds are considered instead of upper bounds, similar results are obtained. For instance, the following theorem can be proved.

Theorem 2

If $a_{ij} \geq 0 \ (i \neq j)$ and if $\underline{F}(t)$ is any function satisfying the relation

$$\underline{F}(t) \leq \underline{Y}[t,\underline{y}(t)] \qquad \text{for} \quad t \geq 0 \quad ,$$

the solutions of the differential equations

$$\dot{\underline{y}} = A\underline{y} + \underline{Y}(t,\underline{y}) \quad , \qquad \underline{y}(0) = \underline{a} \ ,$$

$$\dot{\underline{x}} = A\underline{x} + \underline{F}(t) \quad , \qquad \underline{x}(0) = \underline{a} \ ,$$

are such that

$$\underline{x}(t) \leq \underline{y}(t) \qquad \text{for} \quad t \geq 0 \quad .$$

3. NESTED BOUNDS

Let us assume that, for $t \geq 0$, "a priori" lower and upper bounds $\underline{x}^{(0)}(t)$ and $\underline{z}^{(0)}(t)$ are known for the solution $\underline{y}(t)$ of [3]; those bounds could be obtained, for instance, from simple inspection of the equation. Assuming that $\underline{Y}(t,\underline{y})$ is continuous, it is possible to deduce, from $\underline{x}^{(0)}(t)$ and $\underline{z}^{(0)}(t)$, lower and upper bounds $\underline{F}^{(0)}(t)$ and $\underline{G}^{(0)}(t)$ for the function $\underline{Y}[t,\underline{y}(t)]$:

$$\underline{F}^{(0)}(t) = \inf_{\underline{x}^{(0)}(t) \leq \underline{y}(t) \leq \underline{z}^{(0)}(t)} \underline{Y}[t,\underline{y}(t)] \qquad [10]$$

$$\underline{G}^{(0)}(t) = \sup_{\underline{x}^{(0)}(t) \leq \underline{y}(t) \leq \underline{z}^{(0)}(t)} \underline{Y}[t,\underline{y}(t)] \qquad [11]$$

Because of Corollary 2 and Theorem 2, the solutions $\underline{x}^{(1)}(t)$ and $\underline{z}^{(1)}(t)$ of the equations

$$\underline{\dot{x}}^{(1)} = A\,\underline{x}^{(1)} + \underline{F}^{(0)}(t) \quad , \quad \underline{x}^{(1)}(0) = \underline{a} \quad , \qquad [12]$$

$$\underline{\dot{z}}^{(1)} = A\,\underline{z}^{(1)} + \underline{G}^{(0)}(t) \quad , \quad \underline{z}^{(1)}(0) = \underline{a} \qquad [13]$$

are new bounds for $\underline{y}(t)$, i.e.

$$\underline{x}^{(1)}(t) \leq \underline{y}(t) \leq \underline{z}^{(1)}(t) \qquad \text{for} \quad t \geq 0 \quad .$$

Moreover, if

$$\underline{\dot{x}}^{(0)} \leq A\,\underline{x}^{(0)} + \underline{F}^{(0)}(t) \quad , \quad \underline{x}^{(0)}(0) \leq \underline{a} \quad , \qquad [14]$$

$$\underline{\dot{z}}^{(0)} \geq A\,\underline{z}^{(0)} + \underline{G}^{(0)}(t) \quad , \quad \underline{z}^{(0)}(0) \geq \underline{a} \quad , \qquad [15]$$

the new bounds will lie between the "a priori" ones:

$$\underline{x}^{(0)}(t) \le \underline{x}^{(1)}(t) \le \underline{y}(t) \le \underline{z}^{(1)}(t) \le \underline{z}^{(0)}(t) \quad . \qquad [16]$$

This results from the application of the type of argument used in Theorem 1 and Corollaries 1 and 2 to equations [12] and [14] and to equations [13] and [15].

It must be noticed that, with the conditions [14] and [15], it is no longer necessary to assume that $\underline{x}^{(0)}(t)$ and $\underline{z}^{(0)}(t)$ are bounds for $\underline{y}(t)$, as this results from [16]. Because of the relations [16], new bounds can be found for $\underline{Y}[t,\underline{y}(t)]$ that lie between $\underline{F}^{(0)}(t)$ and $\underline{G}^{(0)}(t)$:

$$\underline{F}^{(1)}(t) = \inf_{\underline{x}^{(1)}(t) \le \underline{y}(t) \le \underline{z}^{(1)}(t)} \underline{Y}[t,\underline{y}(t)] \quad ,$$

$$\underline{G}^{(1)}(t) = \sup_{\underline{x}^{(1)}(t) \le \underline{y}(t) \le \underline{z}^{(1)}(t)} \underline{Y}[t,\underline{y}(t)]$$

Then the solutions $\underline{x}^{(2)}(t)$ and $\underline{z}^{(2)}(t)$ of the equations

$$\dot{\underline{x}}^{(2)} = A\,\underline{x}^{(2)} + \underline{F}^{(1)}(t) \quad , \quad \underline{x}^{(2)}(0) = \underline{a} \quad ,$$

$$\dot{\underline{z}}^{(2)} = A\,\underline{z}^{(2)} + \underline{G}^{(1)}(t) \quad , \quad \underline{z}^{(2)}(0) = \underline{a} \quad ,$$

are new bounds for $\underline{y}(t)$ and satisfy the inequalities

$$\underline{x}^{(1)}(t) \le \underline{x}^{(2)}(t) \le \underline{y}(t) \le \underline{z}^{(2)}(t) \le \underline{z}^{(1)}(t)$$

because

$$\dot{\underline{x}}^{(2)} \ge A\,\underline{x}^{(2)} + \underline{F}^{(0)}(t) \quad ,$$

$$\dot{\underline{z}}^{(2)} \le A\,\underline{z}^{(2)} + \underline{G}^{(0)}(t) \quad .$$

An iterative procedure is thus initiated which, at each step, gives new bounds for the solution $\underline{y}(t)$ of equation [3], the new upper (lower) bound being smaller (larger) than the previous upper (lower) bound. This procedure is explicitly defined by the equations:

$$\dot{\underline{x}}^{(n)} = A\,\underline{x}^{(n)} + \underline{F}^{(n-1)}(t) \quad , \qquad \underline{x}^{(n)}(0) = \underline{a} \quad , \qquad [17]$$

$$\dot{\underline{z}}^{(n)} = A\,\underline{z}^{(n)} + \underline{G}^{(n-1)}(t) \quad , \qquad \underline{z}^{(n)}(0) = \underline{a} \quad , \qquad [18]$$

together with the equations

$$\underline{F}^{(n-1)}(t) = \inf_{\underline{x}^{(n-1)}(t) \leq \underline{y}(t) \leq \underline{z}^{(n-1)}(t)} \underline{Y}[t,\underline{y}(t)] \qquad [19]$$

$$\underline{G}^{(n-1)}(t) = \sup_{\underline{x}^{(n-1)}(t) \leq \underline{y}(t) \leq \underline{z}^{(n-1)}(t)} \underline{Y}[t,\underline{y}(t)] \qquad [20]$$

$$(n = 1,\ 2,\ \ldots.)$$

If the definition of $\underline{F}^{(n-1)}(t)$ is changed to

$$\underline{F}^{(n-1)}(t) = \inf_{\underline{x}^{(n-1)}(t) \leq \underline{y}(t) \leq \underline{z}^{(n)}(t)} \underline{Y}[t,\underline{y}(t)] \quad ,$$

$\underline{z}^{(n)}(t)$ being determined before $\underline{F}^{(n-1)}(t)$, a modified procedure is obtained that will converge more rapidly than the initial one. It must also be noticed that, as long as $\underline{x}^{(0)}(t) \leq \underline{y}(t) \leq \underline{z}^{(0)}(t)$, it is possible to determine a sequence of nested bounds, even if the conditions [14] and [15] are not satisfied. For that purpose, it is sufficient to replace definitions [19] and [20] by

$$\underline{F}^{(n-1)}(t) = \inf_{\underline{\tilde{x}}^{(n-1)}(t) \le \underline{y}(t) \le \underline{\tilde{z}}^{(n-1)}(t)} \underline{Y}[t,\underline{y}(t)] ,$$

$$\underline{G}^{(n-1)}(t) = \sup_{\underline{\tilde{x}}^{(n-1)}(t) \le \underline{y}(t) \le \underline{\tilde{z}}^{(n-1)}(t)} \underline{Y}[t,\underline{y}(t)] ,$$

$$(n = 1,\ 2\ \ldots.)$$

where $\underline{\tilde{x}}^{(n-1)}(t) = \max\ [\underline{x}^{(n-1)}(t),\ \underline{\tilde{x}}^{(n-2)}(t)]$,

$$\underline{\tilde{z}}^{(n-1)}(t) = \min\ [\underline{z}^{(n-1)}(t),\ \underline{\tilde{z}}^{(n-2)}(t)]$$

$$(n = 2,\ 3\ \ldots.)$$

and $\underline{\tilde{x}}^{(0)}(t) = \underline{x}^{(0)}(t)$, $\underline{\tilde{z}}^{(0)}(t) = \underline{z}^{(0)}(t)$.

4. CONVERGENT SUCCESSIVE BOUNDS

The nested bounds of the previous paragraph need not necessarily converge to the solution of equation [3]. However, convergence in this sense can be ascertained if the function $\underline{Y}(t,\underline{y})$ satisfies a Lipschitz condition. Although the notations in the next theorem are those of equations [17] to [20], the result is valid for all the iterative procedures discussed above.

Theorem 3

The sequences $\underline{x}^{(n)}(t)$ and $\underline{z}^{(n)}(t)$, defined by equations [17] to [20], converge to the solution $\underline{y}(t)$ of [3] if a constant L exists such that

$$\| \underline{Y}(t,\underline{y}_1) - \underline{Y}(t,\underline{y}_2) \| \leq L \| \underline{y}_1 - \underline{y}_2 \|$$

for $\underline{y}_1, \underline{y}_2 \in E_n$, $t \geq 0$ ($\| \underline{x} \|$ is any convenient norm defined for $\underline{x} \in E_n$).

PROOF. The sequence of function $\underline{x}^{(n)}(t)$ being non-decreasing and bounded above, converges to a function that will be denoted by $\underline{x}^*(t)$. Similarly, the sequence $\underline{z}^{(n)}(t)$ converges to a function $\underline{z}^*(t)$. Because $\underline{Y}(t,\underline{y})$ is continuous in $\underline{y}$, we have

$$\dot{\underline{x}}^* = A\,\underline{x}^* + \underline{F}^*(t) \quad , \quad \underline{x}^*(0) = \underline{a} \quad ,$$

where $$\underline{F}^*(t) = \inf_{\underline{x}^*(t) \leq \underline{y}(t) \leq \underline{z}^*(t)} \underline{Y}[t,\underline{y}(t)]$$

and $$\dot{\underline{z}}^* = A\,\underline{z}^* + \underline{G}^*(t) \quad , \quad \underline{z}^*(0) = \underline{a}$$

with $$\underline{G}^*(t) = \sup_{\underline{x}^*(t) \leq \underline{y}(t) \leq \underline{z}^*(t)} \underline{Y}[t,\underline{y}(t)] \quad .$$

The difference $\underline{f} = \underline{z}^* - \underline{x}^*$ satisfies the relation

$$\dot{\underline{f}} = A\,\underline{f} + \underline{G}^*(t) - \underline{F}^*(t)\ , \qquad \underline{f}(0) = \underline{0}\ . \qquad [21]$$

But, because of the Lipschitz condition,

$$\underline{G}^*(t) - \underline{F}^*(t) \le L\ \|\,\underline{f}\,\|\ \underline{h}\ ,$$

where $\underline{h}$ is a vector having all components equal to 1; using then Theorem 1 to compare equation [21] and equation

$$\dot{\underline{\lambda}} = A\,\underline{\lambda} + L\ \|\,\underline{\lambda}\,\|\ \underline{h}\ , \qquad \underline{\lambda}(0) = \underline{0}\ . \qquad [22]$$

it is found that

$$\underline{f}(t) \le \underline{\lambda}\,(t) = \underline{0}\ .$$

However, by definition, $\underline{f}(t) \ge \underline{0}$ and, therefore, $\underline{f}(t) = \underline{0}$ or

$$\underline{x}^*(t) = \underline{z}^*(t) = \underline{y}(t)\ .$$

5. APPLICATION TO LANCHESTER-TYPE EQUATIONS

5.1 Example Used

As an illustration of the method discussed above, properties of the solutions of the system

$$\dot{x} = -\alpha\, x - \beta\, x\, y \quad , \qquad x(0) = x_0 > 0 \; , \qquad [22]$$

$$\dot{y} = -\alpha'\, y - \beta'\, x\, y \quad , \qquad y(0) = y_0 > 0 \; , \qquad [23]$$

will be studied; the parameters α, α', β, β' are assumed to be positive.

This system is a variant of models proposed by B.W. Conolly [Ref. 2] to describe situations in naval warfare. The terms $-\alpha\, x$ and $-\alpha'\, y$ could correspond to losses that either cannot be attributed to the opposite side (such as losses due to ship-wrecks or due to obsolescence) or do not, at least within certain limits, depend on the level of the opponent forces appearing explicitly in the model. On the other hand, the terms $-\beta\, x\, y$ and $-\beta'\, x\, y$ could correspond to losses proportional to the number of encounters between units of the forces represented by x and units of the forces represented by y.

The development of the method proposed here is in fact not justified by its application to such a simple problem, but rather by the systematic treatment it provides for a class of differential equations, of which the system [22], [23] is only a simple example. The study of that example will show however which type of properties can be examined by this method.

5.2 Dependence on Parameters and Initial Conditions

A conditional relation will be established between $x(t)$ and $y(t)$ that gives some indication on the effect of the parameters and the initial conditions. In order to establish that relation, the transformation of variables

$$X = \frac{x}{x_0} \quad , \qquad Y = \frac{y}{y_0}$$

is used, the system [22], [23] being rewritten as

$$\dot{X} = -\alpha X - \beta y_0 \; XY \quad , \qquad X(0) = 1,$$

$$\dot{Y} = -\alpha' Y - \beta' x_0 XY \quad , \qquad Y(0) = 1.$$

Using Theorem 1, it is found that

$$X(t) \leq Y(t) \quad \text{or} \quad \frac{x(t)}{x_0} \leq \frac{y(t)}{y_0} \quad \text{for} \quad t \geq 0,$$

if

$$\alpha \geq \alpha' \qquad \text{and} \qquad \beta y_0 \geq \beta' x_0 \quad .$$

Similarly, if $\alpha \geq \alpha'$ the relation

$$\beta y(\tau) \geq \beta' x(\tau) \tag{24}$$

implies the relation

$$\beta y(t) \geq \beta' x(t) \quad \text{for} \quad t \geq \tau$$

Possible values of τ for which [24] is satisfied could be determined using the bounds that will be obtained below.

5.3 Bounds for the Solution

Bounds will now be determined for the solution $x(t)$, $y(t)$ of the system [22], [23]. It appears immediately from the equation that

$$0 \leq x(t) \leq x_0 \quad ,$$

$$0 \leq y(t) \leq y_0 \quad ;$$

from this, it is easy to find improved bounds:

$$x_0 \, e^{-(\alpha + \beta y_0)t} \leq x(t) \leq x_0 \, e^{-\alpha t} \quad , \qquad [25]$$

$$y_0 \, e^{-(\alpha' + \beta' x_0)t} \leq y(t) \leq y_0 \, e^{-\alpha' t} \quad . \qquad [26]$$

Moreover, if

$$x_0 \, e^{-(\alpha + \beta y_0)t} \geq x_1 \quad ,$$

$$y_0 \, e^{-(\alpha' + \beta' x_0)t} \geq y_1 \quad ,$$

i.e., if $\quad t \leq \min \left(\frac{1}{\alpha + \beta y_0} \ln \frac{x_0}{x_1} \, , \; \frac{1}{\alpha' + \beta' x_0} \ln \frac{y_0}{y_1} \right) \qquad [27]$

we have $x_1 \leq x(t) \leq x_0$, $y_1 \leq y(t) \leq y_0$ and, consequently,

$$x_0 \, e^{-(\alpha + \beta y_0)t} \leq x(t) \leq x_0 \, e^{-(\alpha + \beta y_1)t} \quad , \qquad [28]$$

$$y_0 \, e^{-(\alpha' + \beta' x_0)} \leq y(t) \leq y_0 \, e^{-(\alpha' + \beta' x_1)t} \quad . \qquad [29]$$

New bounds have thus been found that are valid as long as t satisfies condition [27]

5.4 Bounds for the Time Required to Reach a Given Level

When studying Lanchester equations, it is important to estimate the time required for some variables to fall below a given value. Bounds for that time are determined here in relation to the example considered. For instant, if τ is defined by $x(\tau) = x_1$, it results from the inequalities [28] that

$$\frac{1}{\alpha + \beta y_0} \ln \frac{x_0}{x_1} \leq \tau \leq \frac{1}{\alpha + \beta y_1} \ln \frac{x_0}{x_1}$$

provided that y_1 satisfies the relation

$$\frac{1}{\alpha' + \beta' x_0} \ln \frac{y_0}{y_1} \geq \frac{1}{\alpha + \beta y_1} \ln \frac{x_0}{x_1} \quad . \qquad [30]$$

In other words, if [30] is satisfied, x reaches the level x_1 before y reaches the level y_1. On the contrary, y reaches y_1 before x reaches x_1 if

$$\frac{1}{\alpha' + \beta' x_1} \ln \frac{y_0}{y_1} \leq \frac{1}{\alpha + \beta y_0} \ln \frac{x_0}{x_1}$$

Consequently, the value $y(x_1)$ taken by y when x equals x_1 is such that

$$\frac{\alpha' + \beta' x_1}{\alpha + \beta y_0} \ln \frac{x_0}{x_1} \leq \ln \frac{y_0}{y(x_1)} \leq \frac{\alpha' + \beta' x_0}{\alpha + \beta y(x_1)} \ln \frac{x_0}{x_1} \quad .$$

5.5 Numerical Determination of Successive Bounds

If $y_o \leq \frac{\alpha'}{\beta}$ and $x_o \leq \frac{\alpha}{\beta'}$, the bounds defined by [25], [26] satisfy the conditions [14], [15]. Starting with those bounds, it is then possible to generate, by iteration, a sequence of nested bounds that can be defined in the following way:

$$x^{(0)}(t) = x_o \, e^{-(\alpha + \beta y_o)t} ,$$

$$y^{(0)}(t) = y_o \, e^{-(\alpha' + \beta' x_o)t} ,$$

$$\dot{x}^{(n)} = -\alpha \, x^{(n)} - \beta \, x^{(n-1)} y^{(n-1)} , \quad x^{(n)}(0) = x_o , \qquad [31]$$

$$\dot{y}^{(n)} = -\alpha' \, y^{(n)} - \beta' \, x^{(n-1)} y^{(n-1)} , \quad y^{(n)}(0) = y_o , \qquad [32]$$

$$(n = 1, 2, 3, \ldots) .$$

The functions $x^{(2n)}(t)$, $y^{(2n)}(t)$ are lower bounds for $x(t)$, $y(t)$, whereas the functions $x^{(2n+1)}(t)$, $y^{(2n+1)}(t)$ are upper bounds. The differential equations [31], [32], can easily be transformed into algebraic equations. Indeed, let us assume that

$$x^{(n-1)}(t) = \sum_{i=1}^{N^{(n-1)}} a_i^{(n-1)} \, e^{-\lambda_i^{(n-1)} t} ,$$

$$y^{(n-1)}(t) = \sum_{i=1}^{N^{(n-1)}} b_i^{(n-1)} \, e^{-\mu_i^{(n-1)} t} ,$$

or that

$$x^{(n-1)}(t) \, y^{(n-1)}(t) = \sum_{j=1}^{\left[N^{(n-1)}\right]^2} c_j^{(n-1)} \, e^{-\nu_j^{(n-1)}(t)} ,$$

where the coefficients $c_j^{(n-1)}$ and exponents $\nu_j^{(n-1)}$ can easily be deduced from $a_i^{(n-1)}$, $b_i^{(n-1)}$, $\lambda_i^{(n-1)}$, $\mu_i^{(n-1)}$ . .

Then, if

$$\nu_j^{(n-1)} \neq \alpha\,,\ \alpha' \quad (\forall_j), \quad \text{we have}$$

$$x^{(n)}(t) = \sum_{j=1}^{N^{(n)}-1} \beta \frac{c_j^{(n-1)}}{\nu_j^{(n-1)} - \alpha} e^{-\nu_j^{(n-1)}(t)} + A^{(n)} e^{-\alpha t} ,$$

$$y^{(n)}(t) = \sum_{j=1}^{N^{(n)}-1} \beta' \frac{c_j^{(n-1)}}{\nu_j^{(n-1)} - \alpha'} e^{-\nu_j^{(n-1)}(t)} + B^{(n)} e^{-\alpha' t}$$

where $N^{(n)} = [N^{(n-1)}]^2 + 1$,

$$A^{(n)} = x_0 - \sum_{j=1}^{N^{(n)}-1} \beta \frac{c_j^{(n-1)}}{\nu_j^{(n-1)} - \alpha} ,$$

$$B^{(n)} = y_0 - \sum_{j=1}^{N^{(n)}-1} \beta' \frac{c_j^{(n-1)}}{\nu_j^{(n-1)} - \alpha'} .$$

The above described procedure has been applied to the system

$$\dot{x} = -0.2\ x - 0.1\ xy, \qquad x(0) = 1 ,$$

$$\dot{y} = -0.2\ x - 0.2\ xy, \qquad y(0) = 1 .$$

Starting with the lower bounds

$$x^{(0)}(t) = e^{-0.3\,t}, \quad y^{(0)}(t) = e^{-0.4\,t} ,$$

three iterations have been performed, the results of which are represented in Figs. 1 and 2. The following relations are shown to hold between the approximate solutions obtained by the second and third iterations:

$$\lim_{t\to\infty} \frac{x^{(2)}(t)}{x^{(3)}(t)} \sim \frac{0.69}{0.72} ,$$

$$\lim_{t\to\infty} \frac{y^{(2)}(t)}{y^{(3)}(t)} \sim \frac{0.38}{0.45} .$$

On the other hand, it can be proved that, for $0 \le t \le 10$, the difference between $x^{(3)}(t)$ and the exact solution $x(t)$ is less than 3×10^{-3}, whereas the difference between $y^{(3)}(t)$ and the exact solution $y(t)$ is less than 6×10^{-3} .

The same method has been applied to the system

$$\dot{x} = 0.001\, x - 0.005\, xy, \qquad x(0) = 1 ,$$

$$\dot{y} = 0.05\, y - 0.02\, xy, \qquad y(0) = 1 ,$$

where the terms $+0.001\,x$ and $+0.05\,y$ could, in biological models, correspond to birth processes. The iterations have been initiated with the lower bounds $x = 0$, $y = 0$ in which case it is not possible to obtain a sequence of nested bounds valid for all $t \ge 0$. However, considering for instance the variable x, it is shown that, after three iterations, an approximation is obtained whose error is less than 3×10^{-3} for $0 \le t \le 10$; the three iterations for x are represented in Fig. 3.

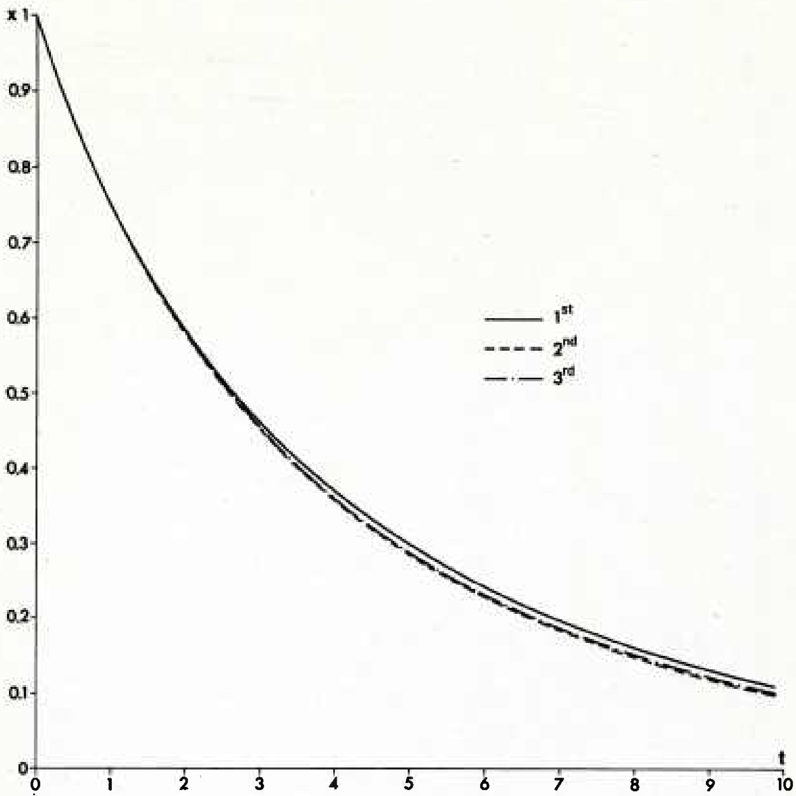

FIG. 1 THREE SUCCESSIVE APPROXIMATIONS FOR THE SOLUTION $x(t)$ OF THE SYSTEM

$$\dot{x} = -0.2x - 0.1xy , \quad x(0) = 1 ,$$
$$\dot{y} = -0.2x - 0.2xy , \quad y(0) = 1 .$$

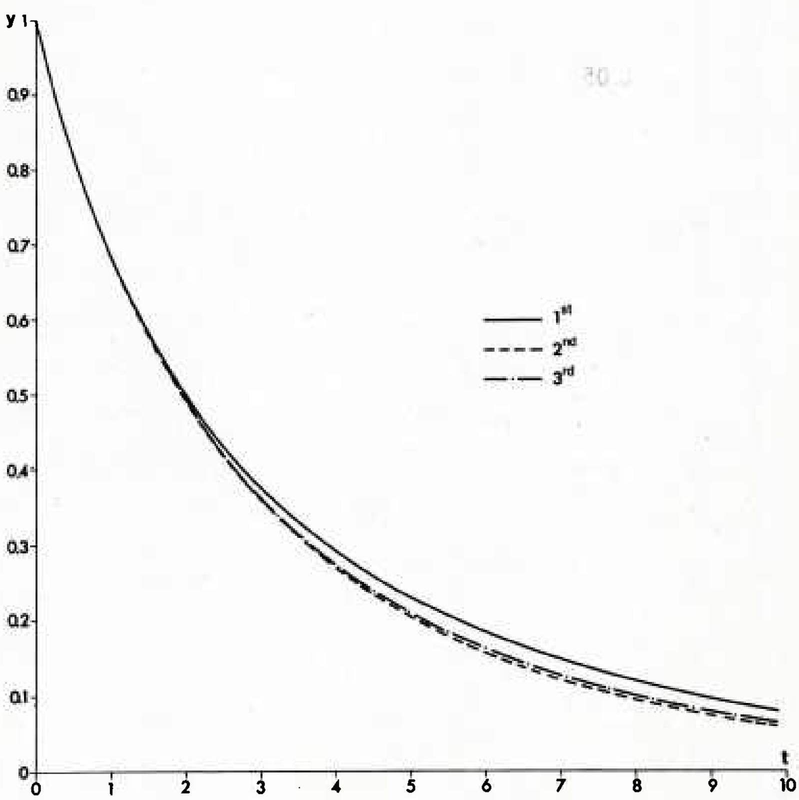

FIG. 2 THREE SUCCESSIVE APPROXIMATIONS FOR THE SOLUTION $y(t)$ OF THE SYSTEM

$$\dot{x} = -0.2x - 0.1xy , \quad x(0) = 1 ,$$
$$\dot{y} = -0.2x - 0.2xy , \quad y(0) = 1 .$$

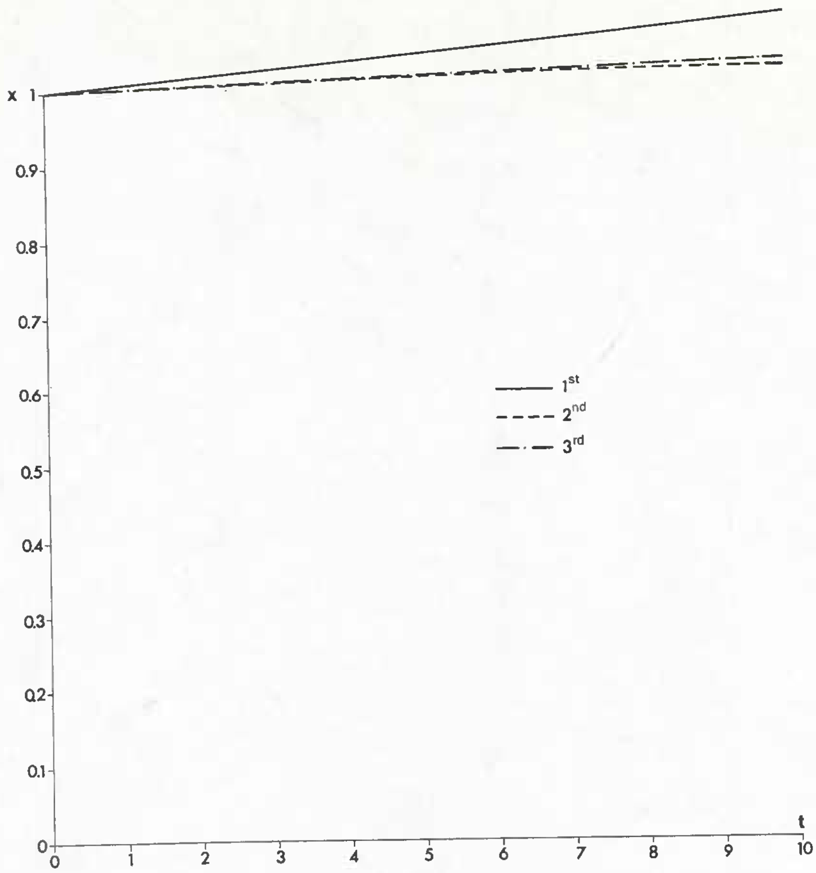

FIG. 3 THREE SUCCESSIVE APPROXIMATIONS FOR THE SOLUTION $x(t)$ OF THE SYSTEM

$$\dot{x} = 0.001x - 0.005xy \quad , \qquad x(0) = 1 \quad ,$$

$$\dot{y} = 0.05y - 0.02xy \quad , \qquad y(0) = 1 \quad .$$

CONCLUSIONS

It has been shown above that the solution of a differential equation could be approximated by successive lower and upper bounds. This integration procedure could be, in principle, applied to any differential equation, provided that "a priori", bounds are known. However, as with the somewhat similar Picard integration method, an appreciable number of iterations can be performed only in favourable cases (essentially when the successive approximations can be expressed as sums of exponential functions).

If our method has rather limited possibilities of use for integration, it offers, on the contrary, a widely applicable approach to the qualitative study of the solutions. Indeed, uniformly valid bounds can provide information concerning the value of the variables at any time and also concerning their asymptotic behaviour. Those bounds, obtained algebraically, will also show the effect of the parameters and initial conditions on the solution.

APPENDIX A

MATRICES WITH POSITIVE NON-DIAGONAL ELEMENTS

An important restriction has been put on the elements of the matrix A in equation [1]; indeed, the non-diagonal elements $a_{ij}(i \neq j)$ have been assumed to be positive. If this is not so, the above described method cannot be applied; however, simple transformations might exist that would transform a given matrix B into a matrix A with positive non-diagonal elements. Simple transformations of a particular type will be considered hereunder, namely transformations that when applied to a vector $y \in E_n$, will change the sign of some components of y, leaving the others unchanged. In other words, we are concerned with transformation matrices $T = \text{diag}(\alpha_1, \ldots, \alpha_n)$ where $\alpha_i = \pm 1$. We want to establish conditions under which such a matrix T exists that transforms a given matrix B into a matrix $A = TBT$ with positive non-diagonal elements, i.e. with elements $a_{ij} \geq 0 \ (i \neq j)$.

<u>Theorem A.1</u> For a given matrix B, a transformation matrix $T = \text{diag}(\alpha_1, \ldots, \alpha_n)$, $(\alpha_i = \pm 1)$, exists that transforms B into a matrix $A = TBT$ with <u>strictly positive</u> non-diagonal elements if, and only if, the elements b_{ij} of B are such that

$$b_{1k}\, b_{k\ell}\, b_{\ell 1} > 0 \qquad \text{for } k \neq 1, \ \ell \neq 1, \ k \neq \ell \ ;$$

$$b_{1k}\, b_{k1} > 0 \ .$$

Necessary Condition : By hypothesis,

$$a_{1k}\, a_{k\ell}\, a_{\ell 1} > 0 \quad \text{for} \quad k \neq 1, \quad \ell \neq 1, \quad k \neq \ell \quad ;$$

consequently

$$\alpha_1^2\, \alpha_k^2\, \alpha_\ell^2\, b_{1k}\, b_{k\ell}\, b_{\ell 1} > 0$$

and $b_{1k}\, b_{k\ell}\, b_{\ell 1} > 0 \quad \text{for} \quad k \neq 1, \quad \ell \neq 1, \quad k \neq \ell$.

Similarly,

$$a_{1k}\, a_{k1} > 0$$

or

$$\alpha_1^2\, \alpha_k^2\, b_{1k}\, b_{k1} > 0$$

Sufficient Condition : Let

$$\alpha_1 = 1 \quad , \quad \alpha_k = \operatorname{sgn}\, (b_{1k}) \quad (k \neq 1)$$

Then,

$$a_{1k} = \operatorname{sgn}\, (b_{1k})\, b_{1k} > 0 \qquad (k \neq 1) \;;$$

$$\begin{aligned} a_{\ell 1} &= \operatorname{sgn}\, (b_{1\ell})\, b_{\ell 1} \\ &= \operatorname{sgn}\, (b_{\ell 1})\, b_{\ell 1} > 0 \qquad (\ell \neq 1) \;; \end{aligned}$$

$$\begin{aligned} a_{k\ell} &= \operatorname{sgn}\, (b_{1k})\, \operatorname{sgn}\, (b_{1\ell})\, b_{k\ell} \\ &= \operatorname{sgn}\, (b_{1k})\, \operatorname{sgn}\, (b_{\ell 1})\, b_{k\ell} > 0 \qquad (k \neq 1,\ \ell \neq 1,\ k \neq \ell) \end{aligned}$$

If we want to find a transformation T such that the non-diagonal elements of $A = T B T$ are positive $(a_{ij} \geq 0,\ i \neq j)$, the conditions

$$b_{jk}\, b_{k\ell}\, b_{\ell j} \geq 0 \quad \text{for} \quad k \neq j,\ \ell \neq j,\ k \neq \ell ,$$

and $\quad b_{jk}\, b_{kj} \geq 0$

are necessary and sufficient.

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